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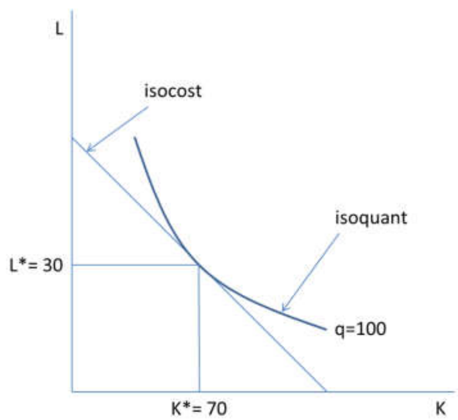
Microeconomic Exercise Cobb-Douglas

The cost minimization problem solved by a firm that behaves according to the principles of microeconomic theory can be represented graphically as



where q is the quantity produced, L and K are the quantities of labor and capital, respectively. The isoquant $q = 100$ represents the combinations of capital and labor which allow to obtain 100 units of product. The isocost graphed represents the minimum cost of producing 100 units of q. At equilibrium, the slope of the isocost - which is given by the ratio of the prices of capital and labor - coincides with the slope of the isoquant - which is given by the ratio of marginal productivities.

The equation for the isocost is

$$C = wL + rK$$

$$L = \frac{C}{w} - \frac{rK}{w}$$

where C is the total cost and w (r) is the remuneration of labor (capital); the slope of the isocost is given by

$$\frac{\partial L}{\partial K} = -\frac{r}{w}$$

Analytically, the cost minimization for a firm that produces with a Cobb-Douglas production function can be written as

$$\min wL + rK$$

$$\text{subject to } q = \gamma L^{\beta_L} K^{\beta_K}$$

where β_L, β_K and γ are the share and efficiency parameters of the Cobb-Douglas production function. We assume constant returns to scale; i.e., $\beta_L + \beta_K = 1$.

The firm chooses how much labor and capital to demand taking as given the prices of the factors, the level of production, and the production technology. The problem of the firm can be rewritten as

$$\min Z = wL + rK + \lambda\left(q - \gamma L^{\beta_L} K^{\beta_K}\right)$$

where λ is the Lagrangian multiplier.

First Order Conditions (FOC)

$$\frac{\partial Z}{\partial L} = w - \lambda\gamma\beta_L L^{\beta_L - 1} K^{\beta_K} = 0$$

$$\frac{\partial Z}{\partial K} = r - \lambda\gamma L^{\beta_L} \beta_K K^{\beta_K - 1} = 0$$

$$\frac{\partial Z}{\partial \lambda} = q - \gamma L^{\beta_L} K^{\beta_K} = 0$$

Manipulating the first FOC,

$$w - \lambda\gamma L^{\beta_L} K^{\beta_K} \beta_L L^{-1} = 0$$

$$w = \frac{\lambda q \beta_L}{L}$$

$$wL = \beta_L \lambda q$$

Manipulating the second FOC,

$$r - \lambda\gamma L^{\beta_L} K^{\beta_K} \beta_K K^{-1} = 0$$

$$r = \frac{\lambda q \beta_K}{K}$$

$$rK = \beta_K \lambda q$$

Then, the demands of factors can also be written as

$$L = \frac{\beta_L \lambda q}{w}$$

$$K = \frac{\beta_K \lambda q}{r}$$

Exercises

(1) How can the Lagrange multiplier λ be interpreted? Prove.

(2) Derive the demand functions for the two goods by solving the following utility maximization problem, which assumes a Cobb-Douglas utility function:

$$\max U = Q_1^{\alpha_1} Q_2^{\alpha_2}$$

$$\text{subject to } y = p_1 Q_1 + p_2 Q_2$$

(3) What are the values of the parameters α_1 and α_2 if the prices p_1 and p_2 are both equal to 1, the consumer's income (y) is 100, and the quantities consumed of goods 1 and 2 are 40 and 60, respectively? HINT: The values of the parameters α_1 and α_2 can be obtained by replacing the above information in the demand functions. (Note: in this exercise we are calibrating the demand functions for goods.)

(4) What are the income elasticities of demand? What are the own-price elasticities of demand? What are the cross-price elasticities of demand?